ELLIPTIC CURVES FROM SEXTICS

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ABSTRACT. Let $\mathcal{N}$ be the moduli space of sextics with 3 (3,4)-cusps. The quotient moduli space $\mathcal{N}/G$ is one-dimensional and consists of two components, $\mathcal{N}_{torus}/G$ and $\mathcal{N}_{gen}/G$. By quadratic transformations, they are transformed into one-parameter families C_s and D_s of cubic curves respectively. First we study the geometry of $\mathcal{N}_\varepsilon/G, \varepsilon = torus, gen$ and their structure of elliptic fibration. Then we study the Mordell-Weil torsion groups of cubic curves C_s over $\mathbf{Q}$ and D_s over $\mathbf{Q}(\sqrt{-3})$ respectively. We show that C_s has the torsion group $\mathbf{Z}/3\mathbf{Z}$ for a generic $s \in \mathbf{Q}$ and it also contains subfamilies which coincide with the universal families given by Kubert [Ku] with the torsion groups $\mathbf{Z}/6\mathbf{Z}$, $\mathbf{Z}/6\mathbf{Z}+\mathbf{Z}/2\mathbf{Z}$, $\mathbf{Z}/9\mathbf{Z}$ or $\mathbf{Z}/12\mathbf{Z}$. The cubic curves D_s has torsion $\mathbf{Z}/3\mathbf{Z}+\mathbf{Z}/3\mathbf{Z}$ generically but also $\mathbf{Z}/3\mathbf{Z}+\mathbf{Z}/6\mathbf{Z}$ for a subfamily which is parametrized by $\mathbf{Q}(\sqrt{-3})$.

1. Introduction

Let $\mathcal{N}_3$ be the moduli space of sextics with 3 (3,4)-cusps as in [O2]. For brevity, we denote $\mathcal{N}_3$ by $\mathcal{N}$. A sextic C is called *of a torus type* if its defining polynomial f has the expression $f(x,y) = f_2(x,y)^3 + f_3(x,y)^2$ for some polynomials f_2, f_3 of degree 2, 3 respectively. We denote by $\mathcal{N}_{torus}$ the component of $\mathcal{N}$ which consists of curves of a torus type and by $\mathcal{N}_{gen}$ the curves of a general type (=not of a torus type). We denote the dual curve of C by C^*. Let $G = \mathrm{PGL}(3,\mathbf{C})$. The quotient moduli space is by definition the quotient space of the moduli space by the action of G.

In §2, we study the quotient moduli space $\mathcal{N}/G$. We will show that $\mathcal{N}/G$ is one dimensional and it has two components $\mathcal{N}_{torus}/G$ and $\mathcal{N}_{gen}/G$ which consist of sextics of a torus type and sextics of a general type respectively. After giving normal forms of these components $C_s, s \in \mathbf{P}^1(\mathbf{C})$ and $D_s, s \in \mathbf{P}^1(\mathbf{C})$, we show that the family C_s *contains a unique sextic* C_{54} *which is self dual* (Theorem 2.8) and C_{54} *has an involution which is associated with the Gauss map* (Proposition 2.12).

In section 3, we study the structure of the elliptic fibrations on the components $\mathcal{N}_\varepsilon/G$, $\varepsilon = torus, gen$ which are represented by the normal families $C_s, s \in \mathbf{P}^1(\mathbf{C})$ and $D_s, s \in \mathbf{P}^1(\mathbf{C})$. Using a quadratic transformation we write these families by smooth cubic curves C_s and D_s which are defined by the following cubic polynomials.

$$C_s: \ x^3 - \frac{1}{4}s(x-1)^2 + sy^2 = 0$$

$$\begin{aligned} D_s: \ & -8x^3 + 1 + (s+35)y^2 - 6x^2 + 3x - 6\sqrt{-3}y - 3\sqrt{-3}x \\ & -6\sqrt{-3}x^2 - 12\sqrt{-3}xy + (s-35)xy = 0 \end{aligned}$$

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We show that $C_s,\ s\in \mathbf{P^1}(\mathbf{C})$ (respectively $D_s,\ s\in\mathbf{P^1}(\mathbf{C})$) *has the structure of rational elliptic surfaces* X_{431} (resp. X_{3333}) in the notation of [Mi-P].

In section 4, we study their torsion subgroups of the Mordell-Weil group of the cubic families C_s and D_s. The family C_s is defined over $\mathbf{Q}$ and D_s is defined over quadratic number field $\mathbf{Q}(\sqrt{-3})$. Both families enjoy beautiful arithmetic properties. We will show that *the torsion group* $(C_s)_{tor}(\mathbf{Q})$ *is isomorphic to* $\mathbf{Z}/3\mathbf{Z}$ *for a generic* $s\in\mathbf{Q}$ *but it has subfamilies* $C_{\varphi_6(u)},\ C_{\varphi_{6,2}(r)},\ C_{\varphi_9(t)}$ *and* $C_{\varphi_{12}(\nu)},\ u,r,t,\nu\in\mathbf{Q}$ *for which the Mordell-Weil torsion group are* $\mathbf{Z}/6\mathbf{Z},\ \mathbf{Z}/6\mathbf{Z}+\mathbf{Z}/2\mathbf{Z},\ \mathbf{Z}/9\mathbf{Z}$ *and* $\mathbf{Z}/12\mathbf{Z}$ *respectively.* Each of these groups is parametrized by a rational function with $\mathbf{Q}$ coefficients which is defined over $\mathbf{Q}$ and this parametrization coincides, up to a linear fractional change of parameter, to the universal family given by Kubert in [Ku].

As for $(D_s)_{tor}(\mathbf{Q}(\sqrt{-3}))$, we show that $(D_s)_{tor}(\mathbf{Q}(\sqrt{-3})$ *is generically isomorphic to* $\mathbf{Z}/3\mathbf{Z}+\mathbf{Z}/3\mathbf{Z}$ *but it also takes* $\mathbf{Z}/3\mathbf{Z}+\mathbf{Z}/6\mathbf{Z}$ *for a subfamily* $D_{\xi_6(t)}$ *parametrized by a rational function with coefficients in* $\mathbf{Q}$ *and defined on* $\mathbf{Q}(\sqrt{-3})$.

2. Normal forms of the moduli $\mathcal{N}$

We consider the submoduli $\mathcal{N}^{(1)}$ of the sextics whose cusps are at $O:=(0,0), A:=(1,1)$ and $B:=(1,-1)$. As every sextic in $\mathcal{N}$ can be represented by a curve in $\mathcal{N}^{(1)}$ by the action of G, we have $\mathcal{N}/G\cong\mathcal{N}^{(1)}/G^{(1)}$ where $G^{(1)}$ is the stabilizer of $\mathcal{N}^{(1)}$: $G^{(1)}:=\{g\in G; g(\mathcal{N}^{(1)})=\mathcal{N}^{(1)}\}$. By an easy computation, we see that $G^{(1)}$ is the semi-direct product of the group $G_0^{(1)}$ and a finite group $\mathcal{K}$, isomorphic to the permutation group $\mathcal{S}_3$ where $G_0^{(1)}$ is defined by

$$G_0^{(1)}:=\Big\{M=\begin{pmatrix} a_1 & a_2 & 0\\ a_2 & a_1 & 0\\ a_1-a_3 & a_2 & a_3\end{pmatrix}\in G; a_3(a_1^2-a_2^2)\neq 0\Big\}$$

Note that $G_0^{(1)}$ is normal in $G^{(1)}$ and $g\in G_0^{(1)}$ fixes singular points pointwise. The isomorphism $\mathcal{K}\cong\mathcal{S}_3$ is given by identifying $g\in\mathcal{K}$ as the permutation of three singular locus O,A,B. We will study the normal forms of the quotient moduli $\mathcal{N}/G\cong\mathcal{N}^{(1)}/G^{(1)}$.

Lemma 2.1. *For a given line* $L:=\{y=bx\}$ *with* $b^2-1\neq 0$, *there exists* $M\in G_0^{(1)}$ *such that* L^M *is given by* $x=0$.

Proof. By an easy computation, the image of L by the action of M^{-1}, where M is as above, is defined by $(a_1-ba_2)y+(a_2-ba_1)x=0$. Thus we take $a_1=ba_2$. Then $a_1^2-a_2^2=a_2^2(b^2-1)\neq 0$ by the assumption. □

Lemma 2.2. *The tangent cone at* O *is not* $y\pm x=0$ *for* $C\in\mathcal{N}^{(1)}$.

Proof. Assume for example that $y-x=0$ is the tangent cone of C at O. The intersection multiplicity of the line $L_1:=\{y-x=0\}$ and C at O is 4 and thus $L_1\cdot C\geq 7$, an obvious contradiction to Bezout theorem. □

Let $\mathcal{N}^{(2)}$ be the subspace of $\mathcal{N}^{(1)}$ consisting of curves $C\in\mathcal{N}^{(1)}$ whose tangent cone at O is given by $x=0$. Let $G^{(2)}$ be the stabilizer of $\mathcal{N}^{(2)}$. By Lemma 2.1 and Lemma 2.2, we have the isomorphism :

Corollary 2.3. $\mathcal{N}^{(1)}/G^{(1)} \cong \mathcal{N}^{(2)}/G^{(2)}$.

It is easy to see that $G^{(2)}$ is generated by the group $G_0^{(2)} := G^{(2)} \cap G_0^{(1)}$ and an element τ of order two which is defined by $\tau(x, y) = (x, -y)$. Note that

$$G_0^{(2)} = \{M = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_1 & 0 \\ a_1 - a_3 & 0 & a_3 \end{pmatrix} \in G_0^{(1)}; \quad a_1 a_3 \neq 0\}$$

For $C \in \mathcal{N}^{(2)}$, we associate complex numbers $b(C), c(C) \in \mathbf{C}$ which are the directions of the tangent cones of C at A, B respectively. This implies that the lines $y-1 = b(C)(x-1)$ and $y+1 = c(C)(x-1)$ are the tangent cones of C at A and B respectively. We have shown that $C \in \mathcal{N}_{torus}^{(2)}$ if and only if $b(C)+c(C) = 0$ and otherwise C is of a general type and they satisfy $c(C)^2 + 3c(C) - b(C)c(C) + 3 - 3b(C) + b(C)^2 = 0$ (§4, [O2]).

We consider the subspaces:

$$\mathcal{N}_{torus}^{(3)} := \{C \in \mathcal{N}_{torus}^{(2)}; b(C) = 0\}, \quad \mathcal{N}_{gen}^{(3)} := \{C \in \mathcal{N}_{gen}^{(2)}; b(C) = c(C) = \sqrt{-3}\}$$

and we put $\mathcal{N}^{(3)} := \mathcal{N}_{torus}^{(3)} \cup \mathcal{N}_{gen}^{(3)}$.

Remark . The common solution of both equations: $b+c = c^2+3c-bc+3-3b+b^2 = 0$ is $(b, c) = (1, -1)$ and in this case, C degenerates into two non-reduced lines $(y^2 - x^2)^2 = 0$ and a conic.

Lemma 2.4. *Assume that $C \in \mathcal{N}^{(2)}$. Then there exists a unique $C' \in \mathcal{N}^{(3)}$ and an element $g \in G^{(2)}$ such that $C^g = C'$. This implies that*

$$\mathcal{N}_{torus}/G \cong \mathcal{N}_{torus}^{(2)}/G^{(2)} \cong \mathcal{N}_{torus}^{(3)}, \quad \mathcal{N}_{gen}/G \cong \mathcal{N}_{gen}^{(2)}/G^{(2)} \cong \mathcal{N}_{gen}^{(3)}$$

Proof. Assume that $C \in \mathcal{N}_{torus}^{(1)}$, $b+c = 0$. Consider an element $g \in G_0^{(1)}$,

$$g^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1-a_3 & 0 & a_3 \end{pmatrix}$$

The image L_A^g is given by $y - x + xa_3 - a_3 - bxa_3 + ba_3 = 0$. Thus we can solve the equation $a_3(1-b) - 1 = 0$ in a_3 uniquely as $a_3 = 1/(1-b)$ as $b \neq 1$. Thus $g \in G_0^{(1)}$ is unique if it fixes the singular points pointwise and thus C' is also unique. It is easy to see that the stabilizer of $\mathcal{N}_{torus}^{(3)}$ is the cyclic group of order two generated by τ, as C' is even in y (see the normal form below) and $C'^{\tau} = C'$ for any $C' \in \mathcal{N}_{torus}^{(3)}$. Thus we have $\mathcal{N}_{torus}^{(2)}/G^{(2)} \cong \mathcal{N}_{torus}^{(3)}$.

Consider the case $C \in \mathcal{N}_{gen}^{(2)}$. Then the images of the tangent cones at A, B by the action of g are given by $y - x + xa_3 - a_3 - bxa_3 + ba_3 = 0$ and $y + x - xa_3 + a_3 - cxa_3 + ca_3 = 0$ respectively. Assume that $b(C^g) = c(C^g)$. Then we need to have $a_3(1-b) - 1 = a_3(-1-c) + 1$, which has a unique solution in a_3, if $(\star)$ $b - c - 2 \neq 0$. Assume that $c^2+3c-bc+3-3b+b^2 = 0$ and $b-c-2 = 0$. Then we get $(b, c) = (1, -1)$ which is excluded as it corresponds to a non-reduced sextic. Thus the condition $(\star)$ is always satisfied. Put $(b', c') := (b(C^g), c(C^g))$. They satisfy the equality $c'^2 + 3c' - b'c' + 3 - 3b' + b'^2 = 0$ and

$b' = c'$. Thus we have either $b' = c' = \sqrt{-3}$ or $b' = c' = -\sqrt{-3}$. However in the second case, $(C^g)^\tau$ belongs to the first case. Thus $b' = c' = \sqrt{-3}$ and $C^g \in \mathcal{N}_{gen}^{(3)}$ as desired. $\square$

2.1. **Normal forms of curves of a torus type.** In [O2], we have shown that a curve in $\mathcal{N}_{torus}^{(1)}$ is defined by a polynomial $f(x,y)$ which is expressed by a sum $f_2(x,y)^3 + sf_3(x,y)^2$ where $f_2(x,y)$ is a smooth conic passing through O, A, B and $f_3(x,y) = (y^2 - x^2)(x-1)$.

Proposition 2.5. *The direction of the tangent cones at O, A and B are the same with the tangent line of the conic $f_2(x,y) = 0$ at these points.*

This is immediate as the multiplicity of $f_3(x,y)^2$ at O, A, B are 4. Assume that $C \in \mathcal{N}_{torus}^{(3)}$, that is, the tangent cones of C at O, A and B are given by $x = 0$, $y - 1 = 0$ and $y + 1 = 0$ respectively. Thus the conic $f_2(x,y) = 0$ is also uniquely determined as $f_2(x,y) = y^2 + x^2 - 2x$. Therefore $\mathcal{N}_{torus}^{(3)}$ is one-dimensional and it has the representation

$$C_s : \quad f_{torus}(x,y,s) := f_2(x,y)^3 + sf_3(x,y)^2 = 0 \tag{2.6}$$

For $s \neq 0, 27, \infty$, C_s is a sextic with three (3,4)-cusps, while C_{27} obtains a node. If $g \in G^{(2)}$ fixes the tangent lines $y \pm 1 = 0$, then $g = e$ or τ. As $C_s^\tau = C_s$, this implies that $C_s^g = C_s$. Thus $C_s \neq C_t$ if $s \neq t$.

2.2. **Normal form of sextics of a general type.** For the moduli $\mathcal{N}_{gen}$ of sextics of a general type, we start from the expression given in §4.1, [O2]. We may assume $b = c = \sqrt{-3}$. Then the parametrization is given by

$$f_{gen}(x,y,s) := f_0(x,y) + sf_3(x,y)^2, \quad f_3(x,y) = (y^2 - x^2)(x-1)$$

where s is equal to a_{06} in [O2] and f_0 is the sextic given by

$$\begin{aligned} f_0(x,y) := {} & y^6 + y^5(6\sqrt{-3} - 6\sqrt{-3}x) + y^4(35 - 76x + 38x^2) \\ & + y^3(-24\sqrt{-3}x + 36\sqrt{-3}x^2 - 12\sqrt{-3}x^3) + y^2(-94x^2 + 200x^3 - 103x^4) \\ & + y(24\sqrt{-3}x^3 - 42\sqrt{-3}x^4 + 18\sqrt{-3}x^5) + 64x^3 - 133x^4 + 68x^5 \end{aligned} \tag{2.7}$$

Let $D_s := \{f_{gen}(x,y,s) = 0\}$ for each $s \in \mathbf{C}$. Observe that $D_0 = \{f_0(x,y) = 0\}$ is a sextic with three (3,4)-cusps and of a general type. For the computation of dual curves using Maple V, it is better to take the substitution $y \mapsto y\sqrt{-3}$ to make the equation to be defined over $\mathbf{Q}$. Summarizing the discussion, we have

Theorem 2.8. *The quotient moduli space $\mathcal{N}/G$ is one dimensional and it has two components.*
(1) The component $\mathcal{N}_{torus}/G$ has the normal forms $C_s = \{f(x,y,s) = 0\}$ where $f(x,y,s) = f_2(x,y)^3 + sf_3(x,y)^2$, $f_2(x,y) = y^2 + x^2 - 2x$ and $f_3(x,y) = (y^2 - x^2)(x-1)$. The curve C_{54} is a unique curve in $\mathcal{N}/G$ which is self-dual.
(2) The component $\mathcal{N}_{gen}/G$ has the normal form: $f_{gen}(x,y,s) = f_0(x,y) + sf_3(x,y)^2$ where f_3 is as above and the sextic $f_0(x,y) = 0$ is contained in $\mathcal{N}_{gen}$. This component has no self-dual curve.

Proof of Theorem 2.8. We need only prove the assertion for the dual curves. The proof will be done by a direct computation of dual curves using the method of §2, [O2] and the above parametrizations. We use Maple V for the practical computation. Here

is the recipe of the proof. Let X^*, Y^*, Z^* be the dual coordinates of X, Y, Z and let $(x^*, y^*) := (X^*/Z^*, Y^*/Z^*)$ be the dual affine coordinates.

(1) Compute the defining polynomials of the dual curves C_s^* and D_s^* respectively, using the method of Lemma 2.4, [O2]. Put $g_{torus}(x^*, y^*, s)$ and $g_{gen}(x^*, y^*, s)$ the defining polynomials.

(2) Let $G_\varepsilon(X^*, Y^*, Z^*, s)$ be the homogenization of $g_\varepsilon(x^*, y^*, s)$, $\varepsilon = torus$ or gen. Compute the discriminant polynomials $\Delta_{Y^*} G_\varepsilon$ which is a homogeneous polynomial in X^*, Z^* of degree 30 (cf. Lemma 2.8, [O1]). Recall that the multiplicity in $\Delta_{Y^*} G_\varepsilon$ of the pencil $X^* - \eta Z^* = 0$ passing through a singular point is generically given by $\mu + m - 1$ where μ is the Milnor number and m is the multiplicity of the singularity ([O2]). Thus the contribution from a (3,4)-cusp is 8. Thus if C_s^* has three (3,4)-cusps, it is necessary that $\Delta_{Y^*}(G) = 0$ has three linear factors with multiplicity ≥ 8.

(3-1) For the curves of a general type, an easy computation shows that it is not possible to get a degeneration into a sextic with 3 (3,4)-cusps by the above reason.

(3-2) For the curves of a torus type, we can see that $s = 54$ is the only parameter such that $C_s^* \in \mathcal{N}$. Thus it is enough to show that $C_{54}^* \cong C_{54}$.

(4) The dual curve C_{54}^* of C_{54} is defined by the homogeneous polynomial

$$\begin{aligned} G(X^*, Y^*, Z^*) := \quad & 128X^{*5}Z^* + 1376X^{*4}Z^{*2} - 192X^{*3}Y{*}^2Z^* + 4664X^{*3}Z^{*3} - 2X^{*2}Y^{*4} \\ & -1584X^{*2}Y^{*2}Z^{*2} + 7090X^{*2}Z^{*4} + 58X^*Y^{*4}Z^* - 3060X^*Y^{*2}Z^{*3} \\ & +5050X^*Z^{*5} + Y^{*6} + 349Y^{*4}Z^{*2} - 1725Y^{*2}Z^{*4} + 1375Z^{*6} \end{aligned}$$

We can see that ${C_{54}}^*$ is isomorphic to C_{54} as $(C_{54}^*)^A = C_{54}$ where

$$A = \begin{pmatrix} -4/3 & 0 & -5/3 \\ 0 & 1 & 0 \\ -5/3 & 0 & -13/3 \end{pmatrix}$$

2.3. **Involution τ on C_{54}.** For a later purpose, we change the coordinates of $\mathbf{P}^2$ so that the three cusps of C_s are at $O_Z := (0,0,1), O_Y := (0,1,0), O_X := (1,0,0)$. A new normal form in the affine space is given by $C_s:\ f_2(x,y)^3 + s f_3(x,y)^2 = 0$ where $f_2(x,y) := xy - x + y$ and $f_3(x,y) := -xy$. In particular, C_{54} is defined by

$$f(x,y) = (xy - x + y)^3 + 54x^2y^2 = 0 \tag{2.9}$$

In this coordinate, C_{54}^* is defined by

$$\begin{aligned} & -28y^3 - 17x^4y^2 - 17x^2y^4 - 28x^3y^3 - 2y^5 + 1788x^3y + 1788x^2y - 17y^4 - 17x^4 \\ & +262xy + 1788x^2y^3 - 1788xy^2 - 262xy^4 + 1788xy^3 - 1788x^3y^2 - 8166x^2y^2 + 28x^3 \\ & +262x^4y - 2x^5y - 2xy^5 + 1 - 17y^2 - 17x^2 + 2x^5 + 2x - 2y + x^6 + y^6 = 0 \end{aligned}$$

It is easy to see that $(C_{54}^*)^{A_1} = C_{54}$ where

$$A_1 := \begin{pmatrix} -1/3 & 7/3 & -1/3 \\ 7/3 & -1/3 & 1/3 \\ -1/3 & 1/3 & -7/3 \end{pmatrix}$$

For a given $A \in \mathrm{GL}(3, \mathbf{C})$, we denote the automorphism defined by the right multiplication of A by φ_A. Let $F(X,Y,Z)$ be the homogenization of $f(x,y)$. Then the Gauss map

$\mathrm{dual}_{C_{54}} : C_{54} \to C_{54}^*$ is defined by

$$\mathrm{dual}_{C_{54}}(X,Y,Z) = (F_X(X,Y,Z), F_Y(X,Y,Z), F_Z(X,Y,Z))$$

where F_X, F_Y, F_Z are partial derivatives. We define an isomorphism $\tau :\ C_{54} \to C_{54}$ by the composition $\varphi_{A_1} \circ \mathrm{dual}_{C_{54}}$. Then τ is the restriction of the rational mapping: $\Psi : \mathbf{C}^2 \to \mathbf{C}^2$, $(x,y) \mapsto (x_d, y_d)$ and

$$\left\{\begin{array}{l} x_d := \frac{-y^3+4x^2-x^2y^3+4x^3y^2-8x^3y-4x^2y^2-8xy-4xy^2-2xy^3+109x^2y+4y^2+4x^3}{-4y^3+x^2-4x^2y^3+4x^3y^2-8x^3y-109x^2y^2-2xy-4xy^2-8xy^3+4x^2y+y^2+4x^3} \\ y_d := -\frac{-4y^3+4x^2-4x^2y^3+x^3y^2-2x^3y-4x^2y^2-8xy-109xy^2-8xy^3+4x^2y+4y^2+x^3}{-4y^3+x^2-4x^2y^3+4x^3y^2-8x^3y-109x^2y^2-2xy-4xy^2-8xy^3+4x^2y+y^2+4x^3} \end{array}\right. \tag{2.10}$$

Observe that τ is defined over $\mathbf{Q}$. C_{54} has three flexes of order 2 at $F_1 := (1, -1/4, 1),\ F_2 := (1/4, -1, 1),\ F_3 := (4, -4, 1)$ and τ exchanges flexes and cusps:

$$\left\{\begin{array}{c} \tau(O_X) = F_1, \tau(O_Y) = F_2, \tau(O_Z) = F_3, \\ \tau(F_1) = O_X, \tau(F_2) = O_Y, \tau(F_3) = O_Z \end{array}\right. \tag{2.11}$$

Furthermore we assert that

Proposition 2.12. *The morphism τ is an involution on C_{54}.*

Proof. By the definition of τ and Lemma 2.13 below, we have ($C := C_{54}$):

$$\tau \circ \tau = (\varphi_{{}^tA_1^{-1}} \circ \mathrm{dual}_C)^2 = (\mathrm{dual}_{C^{A_1}} \circ \varphi_{A_1}) \circ (\varphi_{{}^tA_1^{-1}} \circ \mathrm{dual}_C) = \mathrm{id}$$

as A_1 is a symmetric matrix. □

Let C be a given irreducible curve in $\mathbf{P}^2$ defined by a homogeneous polynomial $F(X,Y,Z)$ and let $B \in \mathrm{GL}(3, \mathbf{C})$. Then C^B is defined by $G(X,Y,Z) := F((X,Y,Z)B^{-1})$.

Lemma 2.13. *Two curves $(C^B)^*$ and $(C^*)^{{}^tB^{-1}}$ coincide and the following diagram commutes.*

$$\begin{array}{ccc} C & \xrightarrow{dual_C} & C^* \\ \downarrow \varphi_B & & \downarrow \varphi_{{}^tB^{-1}} \\ C^B & \xrightarrow{dual_{C^B}} & (C^B)^* \end{array}$$

Proof. The first assertion is the same as Lemma 2, [O2]. The second assertion follows from the following equalities. Let $(a,b,c) \in C$.

$$\begin{aligned} \mathrm{dual}_{C^B}(\varphi_B(a,b,c)) &= (G_X(\varphi_B(a,b,c)), G_Y(\varphi_B(a,b,c)), G_Z(\varphi_B(a,b,c))) \\ &= (F_X(a,b,c), F_Y(a,b,c), F_y(a,b,c))^t B^{-1} = \varphi_{{}^tB^{-1}}(\mathrm{dual}_C(a,b,c) \end{aligned}$$ □

In section 5, we will show that τ is expressed in a simple form as a cubic curve.

3. Structure of elliptic fibrations

We consider the elliptic fibrations corresponding to the above normal forms. For this purpose, we first take a linear change of coordinates so that three lines defined by $f_3(x,y) = 0$ changes into lines $X = 0$, $Y = 0$ and $Z = 0$. The corresponding three cusps are now at $O_Z = (0,0,1), O_Y = (0,1,0), O_X = (1,0,0)$ in $\mathbf{P}^2$. Then we take the quadratic transformation which is a birational mapping of $\mathbf{P}^2$ defined by

$(X,Y,Z) \mapsto (YZ,ZX,XY)$. Geometrically this is the composition of blowing-ups at O_X, O_Y, O_Z and then the blowing down of three lines which are strict transform of $X,Y,Z=0$. It is easy to see that our sextics are transformed into smooth cubics for which $X=0$, $Y=0$ and $Z=0$ are tangent lines of the flex points. Those flexes are the image of the (3,4)-cusps. We take a linear change of coordinates so that the flex on $Z=0$ is moved at $O := (0,1,0)$ with the tangent $Z=0$. Then the corresponding families are described by the families given by $\{h_{torus}(x,y,s)=0; s \in \mathbf{P}^1\}$ and $\{h_{gen}(x,y,s)=0, s\in \mathbf{P}^1\}$ where

$$\begin{cases} C_s: & h_{torus}(x,y,s) := x^3 - \frac{1}{4}s(x-1)^2 + sy^2, \\ D_s: & h_{gen}(x,y,s) := -8x^3 + 1 + (s+35)y^2 - 6x^2 + 3x \\ & \quad -6\sqrt{-3}y - 3\sqrt{-3}x - 6\sqrt{-3}x^2 - 12\sqrt{-3}xy + (s-35)xy \end{cases}$$

Let $H_\varepsilon(X,Y,Z,S,T) = h_\varepsilon(X/Z, Y/Z, S/T)Z^3T$ for $\varepsilon = torus,\ gen$. We consider the elliptic surface associated to the canonical projection $\pi : S_\varepsilon \to \mathbf{P}^1$ where S_ε is the hypersurface in $\mathbf{P}^1 \times \mathbf{P}^2$ which is defined by $H_\varepsilon(X,Y,Z,S,T)=0$.

Case I. Structure of $S_{torus} \to \mathbf{P}^1$. For simplicity, we use the affine coordinate $s = S/T$ of $\{T \neq 0\} \subset \mathbf{P}^1$ and denote $\pi^{-1}(s)$ by C_s. We see that the singular fibers are $s = 0, 27, \infty$. C_∞ consists of three lines, isomorphic to I_3 in Kodaira's notation, [Ko]. C_{27} obtains a node and this fiber is denoted by I_1 in [Ko]. The fiber C_0 is a line with multiplicity 3. The surface S_{torus} has three singular points on the fiber C_0 at $(X,Y,Z) = (0,1/2,1), (0,-1/2,1), (0,1,0)$. Each singularity is an A_2-singularity. We take minimal resolutions at these points. At each point, we need two $\mathbf{P}^1$ as exceptional divisors and let $p : \widetilde{S}_{torus} \to S_{torus}$ be the resolution map. The composition $\widetilde{\pi} := \pi \circ p : \widetilde{S}_{torus} \to \mathbf{P}^1$ is the corresponding elliptic surface. Now it is easy to see that $\widetilde{C_0} := \widetilde{\pi}^{-1}(0)$ is a singular fiber with 7 irreducible components, which is denoted by IV^* in [Ko]. Here we used the following lemma. The elliptic surface $\widetilde{S}_{torus}$ is rational and denoted by X_{431} in [Mi-P].

Assume that the surface $V := \{(s,x,y) \in \mathbf{C}^3; f(s,x,y)=0\}$ has an A_2 singularity at the origin where $f(s,x,y) := sx + y^3 + sx \cdot h(s,x,y)$ where $h(O)=0$. Consider the minimal resolution $\pi : \widetilde{V} \to V$ and let $\pi^{-1}(O) = E_1 \cup E_2$. It is well-known that $E_1 \cdot E_2 = 1$ and $E_i^2 = -2$ for $i=1,2$.

Lemma 3.1. *Consider a linear form $\ell(s,x,y) = as+bx+cy$ and let L' be the strict transform of $\ell = 0$ to $\widetilde{V}$.*
(1) Assume that $b=c=0$ and $a \neq 0$. Then $(\pi^*\ell) = 3L' + 2E_1 + E_2$ and $L' \cdot E_1 = 1$ and L' does not intersect with E_2, under a suitable ordering of E_1 and E_2.
(2) Assume that $abc \neq 0$. Then we have $(\pi^*\ell) = L' + E_1 + E_2$ and $L' \cdot E_i = 1$ for $i=1,2$.

The proof is immediate from a direct computation.

Case II. Structure of $S_{gen} \to \mathbf{P}^1$. Now consider the elliptic surface S_{gen}. Put $D_s = \pi^{-1}(s)$. The singular fibers are at $s = -35, -53+6\sqrt{-3}$, $-53-6\sqrt{-3}$ and $s=\infty$. The fiber $s=\infty$ is already I_3 and S_{gen} is smooth on this fiber. On the other hand, S_{gen} has a A_2-singularity on each fiber D_s, $s = -35, -53+6\sqrt{-3}$, $-53-6\sqrt{-3}$. Let $p : \widetilde{S}_{gen} \to S_{gen}$ be the the minimal resolution map and we consider the composition $\widetilde{\pi} := \pi \circ p : \widetilde{S}_{gen} \to \mathbf{P}^1$

as above. Then using (2) of Lemma 3.1, we see that $\widetilde{\pi} : \widetilde{S}_{gen} \to \mathbf{P}^1$ has four singular fibers and each of them is I_3 in the notation [Ko]. This elliptic surface is also rational and denoted as X_{3333} in [Mi-P].

4. Torsion group of C_s and D_s

Consider an elliptic curve C defined over a number field K by a Weierstrass short normal form: $y^2 = h(x), \quad h(x) = x^3 + Ax + B$. The j-invariant is defined by $j(C) = -1728(4A)^3/\Delta$ with $\Delta = -16(4A^3 + 27B^2)$. We study the torsion group of the Mordell-Weil group of C which we denote by $C_{tor}(K)$ hereafter.

Recall that a point of order 3 is geometrically a flex point of the complex curve C ([Si]) and its locus is defined by $\mathcal{F}(f) := f_{x,x}f_y^2 - 2f_{x,y}f_xf_y + f_{y,y}f_x^2 = 0$ where $f(x,y)$ is the defining polynomial of C ([O1]). In our case, $\mathcal{F}(f) = 24xy^2 - 18x^4 - 12x^2A - 2A$. The unit of the group is given by the point at infinity $O := (0,1,0)$ and the inverse of $P = (\alpha, \beta) \in C$ is given by $(\alpha, -\beta)$ and we denote it by $-P$. For a later purpose, we prepare two easy propositions. Consider a line $L(P, m)$ passing through $-P$ defined by $y = m(x - \alpha) - \beta$. The x-coordinates of two other intersections with C are the solution of $q(x) := f(x, m(x-\alpha) - \beta)/(x - \alpha)$ which is a polynomial of degree 2 in x. Let $\Delta_x q$ be the discriminant of q in x. Note that $\Delta_x q$ is a polynomial in m.

(A) When does a point $Q \in C$ exist such that $2Q = P$.
Assume that a K point $Q = (x_1, y_1)$ satisfies $2Q = P$. Geometrically this implies that the tangent line T_QC passes through $-P$.

Proposition 4.1. *There exists a K-point Q with $2Q = P$ if and only if m is a K-solution of $\Delta_x q(m) = 0$ and x_1 is the multiple solution of $q(x) = 0$. If P is a flex point, $\Delta_x q(m) = 0$ contains a canonical solution which corresponds to the tangent line at P and $m = -f_x(\alpha,\beta)/f_y(\alpha,\beta)$. For any K-solution m with $m \neq -f_x(\alpha,\beta)/f_y(\alpha,\beta)$, the order of Q is equal to $2 \cdot$ order P.*

(B) When does a point $Q \in C$ exist such that $3Q = P$.
Assume that a K-point $Q = (x_1, y_1)$ satisfies $3Q = P$. Put $Q' := 2Q$ and put $Q' = (x_2, y_2)$. Let T_QC be the tangent line at Q. Then T_QC intersects C at $-Q'$. Then $-3Q$ is the third intersection of C and the line L which passes through Q, Q'. Thus three points $-P, Q, Q'$ are colinear. Write L as $y = m(x - \alpha) - \beta$. Then x_1, x_2 are the solutions of $q(x) = 0$. Thus we have

$$x_2 = -\mathrm{coeff}(q,x)/\mathrm{coeff}(q,x^2) - x_1, \quad y_1 = m(x_1 - \alpha) - \beta \tag{4.2}$$

where $\mathrm{coeff}(q, x^i)$ is the coefficient of x^i in $q(x)$. Let $L_Q(x,y)$ be the linear form defining T_QC and let $R(x)$ be the resultant of $f(x,y)$ and $L_Q(x,y)$ in y. Put $R_1(x) := R(-\mathrm{coeff}(q,x)/\mathrm{coeff}(q,x^2) - x)$. Then by the above consideration, $x = x_1$ is a common solution of $q(x) = R_1(x) = 0$. Let $R_2(m)$ be the resultant of $q(x)$ and $R_1(x)$. Note that if $\Delta_x q(m) = 0$, L is tangent to C at Q and $R_2(m) = 0$. In this case, $2Q = P$.

Proposition 4.3. *Assume that there exists a K-point Q with $3Q = P$ and order $Q = 3 \cdot$ order P and let m be as above. Then $R_2(m) = 0$ and $\Delta_x q(m) \neq 0$. Moreover x_1 is given as a common solution of $q(x) = R_1(x) = 0$.*

Actually one can show that $R_2(m)$ is divisible by $(\Delta_x q)^2$.

4.1. **Cubic family associated with sextics of a torus type.** We have observed that the family C_s for $s \in \mathbf{Q}$ is defined over $\mathbf{Q}$. First, recall that C_s is defined by

$$C_s : x^3 - \frac{1}{4}s(x-1)^2 + sy^2 = 0 \tag{4.4}$$

and the Weierstrass normal form is given by $C_s:\ y^2 = x^3 + a(s)x + b(s)$ where

$$a(s) = -\frac{1}{48}s^4 + \frac{1}{2}s^3,\ b(s) = -\frac{1}{24}s^5 + \frac{1}{4}s^4 + \frac{1}{864}s^6 \tag{4.5}$$

Put $\Sigma := \{0, 27, \infty\}$. This corresponds to singular fibers. We have two sections of order 3: $s \mapsto (\frac{1}{12}s^2, \pm\frac{1}{2}s^2)$. Put $P_1 := (\frac{1}{12}s^2, \frac{1}{2}s^2)$. Thus the torsion group is at least $\mathbf{Z}/3\mathbf{Z}$. By [Ma], the possible torsion group which has an element of order 3 is one of $\mathbf{Z}/3\mathbf{Z}$, $\mathbf{Z}/6\mathbf{Z}$, $\mathbf{Z}/2\mathbf{Z}+\mathbf{Z}/6\mathbf{Z}$, $\mathbf{Z}/9\mathbf{Z}$ or $\mathbf{Z}/12\mathbf{Z}$. The j-invariant of C_s is given by

$$j(C_s) := j_{torus}(s), \quad j_{torus}(s) := s(s-24)^3/(s-27) \tag{4.6}$$

(1) Assume that $(C_s)_{tor}(\mathbf{Q})$ has an element of order 6, say $P_2 := (\alpha_2, \beta_2) \in C_s \cap \mathbf{Q}^2$. We may assume that $P_2 + P_2 = P_1$. By Proposition 4.1, this implies that $x = \alpha_2$ must be the multiple solution of

$$q(x) := s^4 - 36s^3 - 72ms^2 - 6xs^2 - 6s^2m^2 + 72m^2x - 72x^2 = 0$$

As $-f_x(-P_1)/f_y(-P_1) = -s/2$, we must have $m \neq -s/2$ and thus

$$\Delta_x' q := \Delta_x q/(2m+s) = s^3 - 32s^2 - 2ms^2 - 4m^2s + 8m^3 = 0 \tag{4.7}$$

The curve $\Delta_x'(q) = 0$ is a rational curve and we can parametrize $\Delta_x' q = 0$ as $s = \varphi_6(u)$, $m = \varphi_6(u)u$ where

$$\varphi_6(u) := 32/(1+2u)(2u-1)^2 \tag{4.8}$$

The point P_2 is parametrized as

$$P_2 = \left(\frac{128}{3}\frac{-1+12u^2}{(2u+1)^2(-1+2u)^4}, \frac{512(6u+1)}{(-1+2u)^5(2u+1)^2}\right) \tag{4.9}$$

where $u \in \mathbf{Q}$. We put $A_6 := \{s = \varphi_6(u); u \in \mathbf{Q}\}$ and $\Sigma_6 := \varphi^{-1}(\Sigma)$. Note that $\Sigma_6 = \{-1/2, 1/2, 5/6, -1/6\}$.

(1-2) Assume that we are given $s = \varphi(u)$ and we consider the case when (4.7) has three rational solutions in m for a fixed s. This is the case if $\varphi_6(u) = \varphi_6(v)$ has two rational solutions different from u. This is also equivalent to $(C_s)_{tor}(\mathbf{Q})$ has $\mathbf{Z}/2\mathbf{Z}+\mathbf{Z}/2\mathbf{Z}$ as a subgroup. The equation is given by the conic

$$Q: \quad 4u^2 - 2u + 4uv - 1 - 2v + 4v^2 = 0 \tag{4.10}$$

By an easy computation, Q is rational and it has a parametrization as follows.

$$u = \varphi_2(r) := \frac{-36+5r^2}{6(12+r^2)}, \quad v(r) := -\frac{1}{6}\frac{(r^2+24r-36)}{(12+r^2)} \tag{4.11}$$

The generators are P_2 of order 6 and $R = (\gamma, 0)$ of order 2 where

$$\gamma := -\frac{81}{4}\frac{(r^4 - 48r^3 + 72r^2 - 432)(12 + r^2)^4}{(r^2 - 36)^4 r^4}$$

Put $\varphi_{6,2}(r) := \varphi_6(\varphi_2(r))$, which is given explicitly as

$$\varphi_{6,2}(r) = 27(12 + r^2)/r^2(r - 6)^2(r + 6)^2$$

We define a subset $A_{6,2}$ of A_6 by the image $\varphi_{6,2}(\mathbf{Q})$. Put $\Sigma_{6,2} := \varphi_{6,2}^{-1}(\Sigma)$. It is given by $\Sigma_{6,2} = \{0, \pm 2, \pm 6\}$.

(2) Assume that there exists a rational point $P_3 = (\alpha_3, \beta_3)$ of order 9 such that $3P_3 = P$. By Proposition 4.3, this is the case if and only if

$$\begin{gathered} R_3(m, s) := 512m^9 + 768m^8 s - 512m^6 s^3 - 1536m^6 s^2 - 192s^4 m^5 \\ -6144m^5 s^3 - 6528m^4 s^4 + 96s^5 m^4 - 12288m^3 s^4 - 2048m^3 s^5 + 64s^6 m^3 + 480s^6 m^2 \\ -15360s^5 m^2 - 6144s^6 m + 384s^7 m - 6s^8 m + 56s^8 - 512s^6 - 768s^7 - s^9 = 0 \end{gathered}$$

has a rational solution. Here R_3 is $R_2/(\Delta_x q)^2(s + 2m)s^4$ up to a constant multiplication. Again we find that the curve $\{(m, s) \in \mathbf{C}^2; R_3(m, s) = 0\}$ is rational and we can parametrize this curve as $s = \varphi_9(t),\ m = \psi_9(t)$ where

$$\left\{ \begin{array}{l} \varphi_9(t) := -\frac{1}{8}\frac{(-1+9t^2-3t+3t^3)^3}{t^3(t-1)^3(t+1)^3} \\ \psi_9(t) := \frac{1}{16}\frac{(-1+9t^2-3t+3t^3)^2(-t+t^3+1+7t^2)}{t^3(t-1)^3(t+1)^3} \end{array} \right. \tag{4.12}$$

The generator $P_3 = (\alpha_3, \beta_3)$ is given by

$$\left\{ \begin{array}{l} \alpha_3 = \frac{1}{768}\frac{(1-18t+15t^2-12t^3+15t^4+30t^5+33t^6)(9t^2-1+3t^3-3t)^4}{(t-1)^6(t+1)^6 t^6} \\ \beta_3 = -\frac{1}{512}\frac{(1+3t^2)(9t^2-1+3t^3-3t)^6}{(t-1)^5(t+1)^7 t^8} \end{array} \right.$$

We put $A_9 := \{\varphi_9(t); t \in \mathbf{Q}\}$ and $\Sigma_9 := \varphi_9^{-1}(\Sigma) = \{0, 1, -1\}$.

(3) Assume that $s \in A_6$ and $(C_s)_{tor}(\mathbf{Q})$ has an element $P_4 = (\alpha_4, \beta_4) \in C_s \cap \mathbf{Q}^2$ of order 12. Then we may assume that $P_4 + P_4 = P_2$. This implies that the tangent line at P_4 passes through $-P_2$. Write this line as $y = n(x - \alpha_2) - \beta_2$. By the same discussion as above, the equality $\Gamma(n_1, u) = 0$ holds where Γ is the polynomial defined by

$$\begin{gathered} \Gamma(u, n_1) := -786432u^4 - 98304n_1 u^3 - 524288u^3 + 393216u^2 - 16384n_1 u^2 \\ -3072n_1^2 u^2 + 131072u + 24576n_1 u + 4096n_1 + 16384 + 256n_1^2 + n_1^4 \end{gathered} \tag{4.13}$$

and $n = n_1/(2u + 1)(2u - 1)^2$. Again we find that $\Gamma = 0$ is a rational curve and we have a parametrization: $u = u(\nu)$ and $n_1 = n_1(\nu)$ where

$$u(\nu) = -\frac{1}{2}\frac{(\nu^4+2\nu^2+5)}{(\nu^4-6\nu^2-3)}, \quad n_1(\nu) = -16\frac{(2\nu^2-4\nu^3-4\nu+\nu^4-3)}{(\nu^4-6\nu^2-3)} \tag{4.14}$$

$$s = \varphi_{12}(\nu) := \varphi_6(u(\nu)), \quad \varphi_{12}(\nu) := -\frac{(\nu^4-3-6\nu^2)^3}{(\nu-1)^4(1+\nu)^4(1+\nu^2)} \tag{4.15}$$

The generator of the torsion group $\mathbf{Z}/12\mathbf{Z}$ is $P_4 = (\alpha_4, \beta_4)$ where

$$\left\{ \begin{array}{l} \alpha_4 := \frac{1}{12}\frac{(\nu^8-12\nu^7+24\nu^6-36\nu^5+42\nu^4+12\nu^3+36\nu-3)(\nu^4-6\nu^2-3)^4}{(\nu-1)^8(\nu+1)^8(\nu^2+1)^2} \\ \beta_4 := -\frac{1}{2}\frac{(\nu^4-6\nu^2-3)^6\nu(\nu^2+3)}{(\nu-1)^7(\nu+1)^{11}(\nu^2+1)^2} \end{array} \right.$$

We put $A_{12} := \{\varphi_{12}(\nu); \nu \in \mathbf{Q}\}$. By definition, $A_{12} \subset A_6$. The singular fibers $\Sigma_{12} := \varphi^{-1}(\Sigma)$ is given by $\{0, \pm 1\}$. Summarizing the above discussion, we get

Theorem 4.16. *The j-invariant is given by $j_{torus}(s) = s(s-24)^3/(s-27)$ and the Mordell-Weil torsion group of C_s is given as follows.*

$$(C_s)_{tor}(\mathbf{Q}) = \begin{cases} \mathbf{Z}/3\mathbf{Z}, & s \in \mathbf{Q} - A_6 \cup A_9 \cup \Sigma \\ \mathbf{Z}/6\mathbf{Z}, & s = \varphi_6(u) \in A_6 - A_{6,2} \cup A_{12},\ u \in \mathbf{Q} - \Sigma_6 \\ \mathbf{Z}/6\mathbf{Z} + \mathbf{Z}/2\mathbf{Z}, & s = \varphi_{6,2}(r) \in A_{6,2},\ r \in \mathbf{Q} - \Sigma_{6,2} \\ \mathbf{Z}/9\mathbf{Z}, & s = \varphi_9(t) \in A_9,\ t \in \mathbf{Q} - \Sigma_9 \\ \mathbf{Z}/12\mathbf{Z}, & s = \varphi_{12}(\nu) \in A_{12},\ \nu \in \mathbf{Q} - \Sigma_{12} \end{cases}$$

4.2. **Comparison with Kubert family.** In [Ku], Kubert gave parametrizations of the moduli of elliptic curves defined over $\mathbf{Q}$ with given torsion groups which have an element of order ≥ 4. His family starts with the normal form:

$$E(b,c) : y^2 + (1-c)xy - by = x^3 - bx^2 \tag{4.17}$$

We first eliminate the linear term of y and then the coefficient of x^2. Let $K_w(b,c)$ be the Weierstrass short normal form, which is obtained in this way. The j-invariant is given by

$$j(E(b,c)) = \frac{(1 - 8bc^2 - 8cb - 4c + 16b + 6c^2 + 16b^2 - 4c^3 + c^4)^3}{b^3(3c^2 - c - 3c^3 - 8bc^2 + b - 20cb + c^4 + 16b^2)}$$

For a given elliptic curve E defined over K with Weierstrass normal form $E : y^2 = x^3 + ax + b$ and a given $k \in K$, the change of coordinates $x \mapsto x/k^2, y \mapsto y/k^3$ changes the normal form into $y^2 = x^3 + ak^4x + bk^6$. We denote this operation by $\Psi_k(E)$.

1. Elliptic curves with the torsion group $\mathbf{Z}/6\mathbf{Z}$. This family is given by a parameter c with $b = c + c^2$.

2. Elliptic curves with the torsion group $\mathbf{Z}/6\mathbf{Z} + \mathbf{Z}/2\mathbf{Z}$. This family is given by a parameter c_1 with $b = c + c^2$ and $c = (10 - 2c_1)/(c_1^2 - 9)$.

3. Elliptic curves with the torsion group $\mathbf{Z}/9\mathbf{Z}$. The corresponding parameter is f and $b = cd, c = fd - f, d = f(f-1) + 1$.

4. Elliptic curves with the torsion group $\mathbf{Z}/12\mathbf{Z}$. The corresponding parameter is τ and $b = cd, c = fd - f, d = m + \tau, f = m/(1-\tau)$ and $m = (3\tau - 3\tau^2 - 1)/(\tau - 1)$.

Proposition 4.18. *Our family $C_{\varphi_6(u)}, C_{\varphi_{6,2}(r)}, C_{\varphi_9(t)}, C_{\varphi_{12}(\nu)}$ are equivalent to the respective Kubert families. More explicitly, we take the following change of parameters to make their j-invariants coincide with those of Kubert and then we take the change of coordinates of type Ψ_k to make the Weierstrass short normal forms to be identical with $K_w(x,y)$.*

1. *For $C_{\varphi_6(u)}$, take $u = -(c-1)/2(3c+1)$ and $k = c^2(c+1)/(3c+1)^2$.*
2. *For $C_{\varphi_{6,2}(r)}$, take $r = -12/(c_1 - 3)$ and $k = 4(-5+c_1)^2(c_1-1)^2/(c_1^2 - 6c_1 + 21)^2$ $/(c_1 - 3)(c_1 + 3)$.*
3. *For $C_{\varphi_9(t)}$, take $t = -f/(f-2)$ and $k = f^3(f-1)^3/(f^3 - 3f^2 + 1)^2$.*
4. *For $C_{\varphi_{12}(\nu)}$, take $\nu = -1/(2\tau - 1)$ and $k = (\tau - 1)\tau^4(-2\tau + 2\tau^2 + 1)(-1 + 2\tau)^2/$ $(6\tau^4 - 12\tau^3 + 12\tau^2 - 6\tau + 1)^2$.*

We omit the proof as the assertion is immediate from a direct computation.

4.3. **Involution on C_{54}.** We consider again the self dual curve $C := C_{54}$ (see §3). The Weierstrass normal form is $y^2 = x^3 - 98415x + 11691702$. Note that $54 \in A_6 - A_{12} \cup A_{6,2} \cup \Sigma$. In fact, $54 = \varphi_6(1/6)$ and $54 \notin A_{12} \cup A_{6,2}$. The j-invariant is 54000 and the torsion group $C_{tor}(\mathbf{Q})$ is $\mathbf{Z}/6\mathbf{Z}$ and the generator is given by $P = (-81, 4374)$. Other rational points are $2P = (243, -1458), 3P = (162, 0), 4P = (243, 1458), 5P = (-81, -4374)$, and $O = (0, 1, 0)$ (= the point at infinity). Recall that C has an involution τ which is defined by (2.10) in §3. To distinguish our original sextic and cubic, we put

$$C^{(6)} : (xy - x + y)^3 + 54x^2y^2 = 0, \quad C^{(3)} : y^2 = x^3 - 98415x + 11691702$$

The identification $\Phi : C^{(3)} \to C^{(6)}$ is given by the rational mapping:

$$\Phi(x, y) = (-2916/(27x - 5103 - y), 2916/(y + 27x - 5103))$$

and the involution $\tau^{(3)}$ on $C^{(3)}$ is given by the composition $\Phi^{-1} \circ \tau \circ \Phi$. After a boring computation, $\tau^{(3)}$ is reduced to an extremely simple form in the Weierstrass normal form and it is given by $\tau^{(3)}(x, y) = (p(x, y), q(x, y))$ where

$$p(x,y) := 81\frac{2x - 567}{x - 162} \quad q(x,y) := -19683\frac{y}{(x - 162)^2} \tag{4.19}$$

Note that C has another canonical involution ι which is an automorphism defined by $\iota : (x, y) \mapsto (x, -y)$. We can easily check that $\tau^{(3)} \circ \iota = \iota \circ \tau^{(3)}$. Note that $\tau^{(3)}(P) = 2P, \tau^{(3)}(2P) = P, \tau^{(3)}(3P) = O, \tau^{(3)}(O) = 3P, \tau^{(3)}(4P) = 5P, \tau^{(3)}(5P) = 4P$. Let $\eta : C \to C$ be the translation by the 2-torsion element $3P$ i.e., $\eta(x, y) = (x, y) + (162, 0)$. It is easy to see that $\tau^{(3)}$ is the composition $\iota \circ \eta$. That is $\tau^{(3)}(x, y) = (x, -y) + (162, 0)$ where the addition is the addition by the group structure of C_{54}. Thus

Theorem 4.20. *The involution τ on sextics $C^{(6)}$ is equal to the involution $\tau^{(3)}$ on $C^{(3)}$ which is defined by (4.19) and it is also equal to $(x, y) \mapsto (x, -y) + (162, 0)$.*

4.4. **Cubic family associated with sextics of a general type.** We consider the family of elliptic D_s curves associated to the moduli of sextics of a general type with three (3,4)-cusps. Recall that D_s is defined by the equation:

$$\begin{aligned} D_s : \quad & -8x^3 + 1 + sy^2 + 35y^2 - 6x^2 + 3x - 6\sqrt{-3}y - 3\sqrt{-3}x \\ & -6\sqrt{-3}x^2 - 12\sqrt{-3}xy + (s - 35)xy = 0 \end{aligned}$$

This family is defined over $\mathbf{Q}(\sqrt{-3})$. We change this polynomial into a Weierstrass normal form by the usual process killing the coefficient of y and then by killing the coefficient of x^2. A Weierstrass normal forms is given by $y^2 = x^3 + a(s)x + b(s)$ where

$$\begin{cases} a(s) := -\frac{1}{768}(s + 47)(s + 71)(s^2 + 70s + 1657) \\ b(s) := \frac{1}{55296}(s^2 + 70s + 793)(s^4 + 212s^3 + 17502s^2 \\ \qquad\qquad +648644s + 9038089) \end{cases} \tag{4.21}$$

The singular fibers are $s = -35, -53 + 6\sqrt{-3}, -53 - 6\sqrt{-3}$ and $s = \infty$. Put $\Sigma = \{-35, -53 + \pm 6\sqrt{-3}, \infty\}$. In this section, we consider the Modell-Weil torsion over the

quadratic number field $\mathbf{Q}(\sqrt{-3})$. First we observe that this family has 8 sections of order three $\pm P_{3,i}, i = 1, \dots, 4$ where $P_{3,i}$ are given by

$$P_{3,1} := (x_{3,1}, y_{3,1}), \quad \begin{cases} x_{3,1} := 5041/48 + 71s/24 + s^2/48 \\ y_{3,1} := 2917/4 + 53s/2 + s^2/4 \end{cases} \tag{4.22}$$

$$P_{3,2} := (x_{3,2}, y_{3,2}) \begin{cases} x_{3,2} := -2209/16 - 47s/8 - s^2/16 \\ y_{3,2} := \sqrt{-3}(s^2 + 106s + 2917)(s + 35)/144 \end{cases} \tag{4.23}$$

$$P_{3,3} := (x_{3,3}, y_{3,3}), \quad \begin{cases} x_{3,3} := s^2/48 + 793/48 + 35s/24 + (s + 35)\sqrt{-3}/2 \\ y_{3,3} := (-1 + \sqrt{-3})(s + 35)(s + 6\sqrt{-3} + 53)/8 \end{cases} \tag{4.24}$$

$$P_{3,4} := (x_{3,4}, y_{3,4}), \quad \begin{cases} x_{3,4} := s^2/48 + 793/48 + 35s/24 - (s + 35)\sqrt{-3}/2 \\ y_{3,4} := -(1 + \sqrt{-3})(s + 53 - 6\sqrt{-3}(s + 35)/8 \end{cases} \tag{4.25}$$

Thus they generate a subgroup isomorphic to $\mathbf{Z}/3\mathbf{Z} + \mathbf{Z}/3\mathbf{Z}$. We can take the generators $P_{3,1}, P_{3,2}$ for example. Thus by [Ke-Mo], $(D_s)_{tor}(\mathbf{Q}(\sqrt{-3}))$ is isomorphic to one of the following.

(a) $\mathbf{Z}/3\mathbf{Z} + \mathbf{Z}/3\mathbf{Z}$, (b) $\mathbf{Z}/3\mathbf{Z} + \mathbf{Z}/6\mathbf{Z}$ and (c) $\mathbf{Z}/6\mathbf{Z} + \mathbf{Z}/6\mathbf{Z}$.

The case (b) is forgotten in the list of [Ke-Mo] by an obvious type mistake. By the same discussion as in 5.1, there exists $P \in D_s$ with order 6 and $2P = P_{3,1}$ if and only if

$$\begin{aligned} \Delta(s, m) := s^3 + 85s^2 - 4ms^2 - 568ms + 1555s - 16m^2 s - 1136m^2 \\ -15465 - 20164m + 64m^3 = 0 \end{aligned}$$

Fortunately the variety $\Delta = 0$ is again rational and we can parametrize it as

$$s = \xi_6(t), \quad \xi_6(t) := -(27t^3 - 1304t^2 + 17920t - 71680)/(t - 8)(t - 16)^2 \tag{4.26}$$

$$m = \psi(t), \quad \psi(t) := -(-128t^2 + 3t^3 + 1536t - 6144)/(t - 8)(t - 16)^2 \tag{4.27}$$

It turns out that the condition for the existence of $Q \in D_s$ with $2Q = P_{3,2}$ is the same with the existence of P, $2P = P_{3,1}$. Assume that $s = \xi_6(t)$. Then by an easy computation, we get $P = (x_{6,1}, y_{6,1})$ and $Q = (x_{6,2}, y_{6,2})$ where

$$x_{6,1} := -\tfrac{1}{3}\tfrac{(-3072t^5 + 11796480t^2 + 86016t^4 - 1327104t^3 - 56623104t + 113246208 + 47t^6)}{(t-8)^2(t-16)^4}$$

$$y_{6,1} := \tfrac{-4t^3(t^2 - 24t + 192)(7t^2 - 144t + 768)}{(t-16)^5(t-8)^2}$$

$$x_{6,2} := \tfrac{1}{3}\tfrac{(37t^6 - 2016t^5 + 40704t^4 - 294912t^3 - 1179648t^2 + 28311552t - 113246208)}{(t-8)^2(t-16)^4}$$

$$y_{6,2} := -\tfrac{8}{7}\tfrac{\sqrt{-3}(t-12)(t-12-4\sqrt{-3})(7t-72+8\sqrt{-3})(7t-72-8\sqrt{-3})t(t-12+4\sqrt{-3})}{(t-16)^3(t-8)^3}$$

It is easy to see by a direct computation that $3P = 3Q = (\alpha, 0)$ where

$$\alpha := -\frac{2}{3}\frac{(t^2 - 48t + 384)(13t^4 - 528t^3 + 8064t^2 - 55296t + 147456)}{(t - 8)^2(t - 16)^4}$$

and $Q - P = P_{3,3}$. Now we claim that

Claim 1. $(D_s)_{tor}(\mathbf{Q}(\sqrt{-3})) = \mathbf{Z}/3\mathbf{Z} + \mathbf{Z}/6\mathbf{Z}$ with generators $P_{3,3}$ and P.

In fact, if the torsion is $\mathbf{Z}/6\mathbf{Z}+\mathbf{Z}/6\mathbf{Z}$, there exist three elements of order two. However $f_0(x) := f(x,0)$ factorize as $(x-\alpha)f_{0,0}(x)$ and their discriminants are given by

$$\Delta_x f_0 := \tfrac{2048t^6(t-12)^3(t^2-24t+192)^3(7t^2-144t+768)^6}{(t-8)^9(t-16)^{18}}$$
$$\Delta_x f_{0,0} := 165888(t-12)^3(t^2-24t+192)^3(t-8)^7(t-16)^8$$

Consider quartic $Q_4 : g(t,v) := 165888(t-12)(t^2-24t+192)(t-8)-v^2=0$. Thus D_s has three two torsion elements if and only if the quartic $g(t,v)=0$ has $\mathbf{Q}(\sqrt{-3})$-point (t_0,v_0) with $t_0 \neq 8, 16, 12, 12\pm 4\sqrt{-3}$. The proof of Claim is reduces to:

Assertion 1. There are no such point on Q_4.

Proof. By an easy birational change of coordinates, $g(t,v)=0$ is equivalent to the elliptic curve $C := \{x^3+1/16777216-y^2=0\}$. We see that C has two element of order three, $(0,\pm 1/4096)$ and three two-torsion $(-1/256,0), (1/512-1/512\sqrt{-3},0)$ and $(1/512+1/512\sqrt{-3},0)$. Again by [Ke-Mo], $C_{tor}(\mathbf{Q}(\sqrt{-3}))=\mathbf{Z}/2\mathbf{Z}+\mathbf{Z}/6\mathbf{Z}$. As the rank of C is 0 ([S-Z]), there are exactly 12 points on C. They correspond to either zeros or poles of $\Delta_x(f_0)$. This implies that the quartic Q_4 has no non-trivial points and thus C does not have three 2-torsion points. This completes the proof of the Assertion and thus also proves the Claim. □

Now we formulate our result as follows. Let $A_6 = \{s=\xi_6(t); t\in \mathbf{Q}(\sqrt{-3})\}$ and $\Sigma_6 := \xi_6^{-1}(\Sigma)$ is given by $\Sigma_6=\{8,16,0,12,12\pm 4\sqrt{-3},(72\pm 8\sqrt{-3})/7\}$.

Theorem 4.28. *The Mordell-Weil torsion of D_s is given by*

$$(D_s)_{tor}(\mathbf{Q}(\sqrt{-3})) = \begin{cases} \mathbf{Z}/3\mathbf{Z}+\mathbf{Z}/3\mathbf{Z} & s\in \mathbf{Q}(\sqrt{-3})-A_6\cup\Sigma \\ \mathbf{Z}/6\mathbf{Z}+\mathbf{Z}/3\mathbf{Z} & s=\xi_6(t)\in A_6,\ t\in\mathbf{Q}(\sqrt{-3})-\Sigma_6 \end{cases}$$

The j-invariant is given by

$$j(D_s)=\frac{1}{64}\frac{(s+47)^3(s+71)^3(s^2+70s+1657)^3}{(s+35)^3(s^2+106s+2917)^3}$$

4.5. **Examples.** (A) First we consider the case of elliptic curves C_s. In the following examples, we give only the values of parameter s as the coefficients are fairly big. The corresponding Weierstrass normal forms are obtained by (4.5).

1. $s=54$. The curve C_{54} with torsion group $\mathbf{Z}/6\mathbf{Z}$ has been studied in §4.3.

2. Take $r=3,\ s=\varphi_{6,2}(3)=343/9$. Then the torsion group is isomorphic to $\mathbf{Z}/6\mathbf{Z}+\mathbf{Z}/2\mathbf{Z}$ with generators $P_2=(-55223/972,-588245/486)$ and $R=(88837/972,0)$. The j-invariant is given by $7^3\cdot 127^3/2^2\cdot 3^6\cdot 5^2$.

3. Take $t=-3,\ s=\varphi_9(-3)=1/216$. Then the torsion group is isomorphic to $\mathbf{Z}/9\mathbf{Z}$ and the generator $P_3=(289/559872,-7/419904)$. The j-invariant is $71^3\cdot 73^3/2^9\cdot 3^9\cdot 7^3\cdot 17$.

4. Take $\nu=3,\ s=\varphi_{12}(3)=-27/80$. Then the torsion is isomorphic to $\mathbf{Z}/12\mathbf{Z}$ with generator $P_4=(-2997/25600,-6561/102400)$. The j-invariant is $-11^3\cdot 59^3/2^{12}\cdot 3\cdot 5^3$.

(B) We consider elliptic curves D_s defined over $\mathbf{Q}(\sqrt{-3})$. The normal form is given by (4.21).

5. Take $s=1$. Then $(D_1)_{tor}(\mathbf{Q}(\sqrt{-3}))=\mathbf{Z}/3\mathbf{Z}+\mathbf{Z}/3\mathbf{Z}$ and the generators are $(x_{3,1},y_{3,1})=(108,756)$ and $(x_{3,2},y_{3,2})=(-144,756\sqrt{-3})$. The j-invariant is $2^{15}3^3/7^3$.

6. Take $t = 4$ and $s = -299/9$. Then the torsion is isomorphic to $\mathbf{Z}/6\mathbf{Z} + \mathbf{Z}/3\mathbf{Z}$. The generators can be taken as $(x_{6,1}, y_{6,1}) = (-2351/243, -532/243)$ and $(x_{3,3}, y_{3,3}) = (8\sqrt{-3}/9 - 2171/243, -680/81 + 248\sqrt{-3}/81)$. The j-invariant is given by $5^3 \cdot 17^3 \cdot 31^3 \cdot 2203^3/2^6 \cdot 3^6 \cdot 7^3 \cdot 19^6$.

4.6. **Appendix. Parametrization of rational curves.** Parametrizations of a rational curves are always possible and there exists even some programs to find a parametrization on Maple V. For the detail, see [Ab-Ba] and [vH] for example. In our case, it is easy to get a parametrization by a direct computation. For a rational curves with degree less than or equal four is easy. For other case, we first decrease the degree, using suitable bitational maps. We give a brief indication. We remark here that the parametrization is unique up to a linear fractional change of the parameter.

(1) For the parametrization of $s^3 - 32s^2 - 2ms^2 - 4m^2s + 8m^3 = 0$, put $m = us$.

(2) For the parametrization of

$$R_3(m, s) := 512m^9 + 768m^8s - 512m^6s^3 - 1536m^6s^2 - 192s^4m^5$$
$$-6144m^5s^3 - 6528m^4s^4 + 96s^5m^4 - 12288m^3s^4 - 2048m^3s^5 + 64s^6m^3 + 480s^6m^2$$
$$-15360s^5m^2 - 6144s^6m + 384s^7m - 6s^8m + 56s^8 - 512s^6 - 768s^7 - s^9 = 0$$

put successively $s = s_1/m_1$ and $m = 1/m_1$, then put $n_1 = n_2/s_1^2$, then $s_1 = s_2 - 2$ and $n_2 = n_4s_2$. This changes degree of our curve to be 6. Then $s_2 + s_3 - 4$ and $n_4 = n_5 + 2$ and $n_5 = n_6s_3$. This changes our curve into a quartic. Other computation is easy.

4.7. **Further remark.** Professor A. Silverberg kindly communicated us about the paper [R-S]. He gave a universal family for $\mathbf{Z}/3\mathbf{Z} + \mathbf{Z}/3\mathbf{Z}$ over $\mathbf{Q}(\sqrt{-3})$, which is given by $A(u) : y^2 = x^3 + a_0(u)x + b_0(u)$ where

$$a_0(u) = -27u(8 + u^3), \quad b_0(u) = -54(8 + 20u^3 - u^6)$$

and the subfamily, given by $u = (4 + \tau^3)/(3\tau^2)$, describes elliptic curves with torsion $\mathbf{Z}/6\mathbf{Z} + \mathbf{Z}/3\mathbf{Z}$. Again by an easy computation, we can show that by the change of parameter $s = -47 + 12u$ we can identify D_s and $A(u)$. Our subfamily for $\mathbf{Z}/6\mathbf{Z} + \mathbf{Z}/3\mathbf{Z}$ is also the same with that of [R-S] by the fractional change of parameter: $t = 8(\tau - 2)/(\tau - 1)$.

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